Extended self-similarity in numerical simulations of three-dimensional anisotropic turbulence

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Using a code based on the lattice Boltzmann equation, we have performed numerical simulations of a turbulent shear flow. We investigate the scaling behavior of the structure functions in presence of *anisotropic* homogeneous turbulence, and we show that although extended self-similarity does not hold when strong shear effects are present, a more generalized scaling law can still be defined. [S1063-651X(96)50706-3]

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In the last few years there has been a growing attention on the scaling properties of fully developed turbulence and, in particular, on the characterization of the probability distribution function of the velocity increments $\delta_r v \equiv v_x(\mathbf{x}+\mathbf{r}) - v_x(\mathbf{x})$, i.e. the velocity difference in the x direction between two points at distance r.

To this aim, usually one considers the scaling properties of the structure functions defined as

$$F_n(r) = \langle |\delta_r v|^n \rangle. \quad (1)$$

According to the Kolmogorov theory [1] a scaling law for (1) is expected to hold in the so-called inertial range, $\eta \ll r \ll L$ (L being the integral scale of the flow and η the Kolmogorov scale):

$$F_n(r) = A_n(\epsilon r)^{n/3} \quad (2)$$

where A_n are dimensionless constants and ϵ is the mean rate of energy dissipation.

There have been many experimental and numerical results suggesting that, because of the intermittency of the velocity gradients, the relation (2) is violated, giving an anomalous scaling law with scaling exponents $\zeta_n \neq n/3$. By taking into account the fluctuations of the energy dissipation field, Eq. (2) has been modified by Kolmogorov [2], who introduced the refined similarity hypothesis (RSH):

$$F_n(r) = A'_n \langle \epsilon_r^{n/3} \rangle r^{n/3} \quad (3)$$

where ϵ_r is the local rate of energy transfer,

$$\epsilon_r \equiv \frac{1}{r^3} \int_{B(r)} \epsilon(\mathbf{x}) d^3x.$$

At present, most of the efforts, both theoretical and experimental, are devoted to the determination of the anomalous scaling exponents and to the investigation of the role played by the RSH.

The aim of this work is to investigate the scaling properties of the structure functions in the case of a homogeneous shear flow, as a simple example of anisotropic homogeneous turbulence. We are mainly interested in studying the scaling laws of the structure functions and we want to establish if the extended self-similarity (ESS), recently introduced in the literature [3–5], still holds for shear flows, i.e., in the presence of a nonisotropic turbulent flow.

In this paper we first remind the reader of some concepts about ESS and its relevance in order to estimate the ζ_n. Next we briefly describe the shear flows and some of their properties. Finally we discuss the numerical simulation and show that ESS does not hold for shear flows, while a generalized scaling law, involving both ESS and RSH, is valid.

In principle, we can determine the scaling exponents ζ_n by means of experimental and numerical measures, but in the latter case some technical problems arise.

We define, as usual, the Reynolds number as $\mathrm{Re} = UL/\nu$, where U is the typical velocity of the flow, L is a typical macroscopic scale in the system, and ν is the kinematic viscosity. The highest Reynolds numbers that can be achieved by laboratory experiments are about 10^6, 10^7, while numerical simulations performed with the most powerful computers now available cannot reach these limits. As the computational effort grows like Re^3, it could seem very hard to obtain good estimates, at least comparable to the experimental results, of the scaling exponents by the numerical simulations. The concept of ESS can help us to fill this gap.

The idea is to investigate the scaling behavior of one structure function against the other, namely,

$$F_n(r) \sim F_m(r)^{\beta(n,m)}. \quad (4)$$

In particular, it is expected that, at least in the inertial range, $\beta(n,3) = \zeta_n$. Actually, there is strong evidence that ESS is a powerful tool to investigate the scaling laws and that it has many advantages in respect to the usual scaling against r. Namely, it holds both for the dissipative range $r \sim (4–5)\eta$ and also for low Reynolds numbers. Finally, the two previous properties allow a very accurate determination of the scaling exponents. Indeed, the ζ_n can be estimated with an error of just a few percent.

The above statements can be summarized as follows. We can always write the structure functions in the following way:

$$F_p(r) = C_p U_0^p \left[\frac{r}{L} f_p\left(\frac{r}{\eta} \right) \right]^{\zeta_p} \quad (5)$$

with $U_0^3 = F_3(r)$, $L = U_0^3/\epsilon$ being the integral scale, and C_p dimensionless constants selected in such a way that $f_p(r/\eta) = 1$ for $r \gg \eta$. ESS implies that, for all the orders p, the function $f_p(r/\eta) \equiv f(r/\eta)$ is the same.

We want to understand the effects of the lack of isotropy on the anomalous scaling law defined in (4). To this effect, we consider a simple shear flow.

Let us consider the usual Navier-Stokes equations describing a viscous, incompressible fluid of density ρ, and velocity field $\mathbf{v}(\mathbf{x},t)$:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \Delta \mathbf{v} + \mathbf{f},$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0. \tag{6}$$

Let us indicate the stationary solution of the above equations as $\mathbf{U}$, and define the turbulent velocities $\mathbf{w}$ as $\mathbf{v} = \mathbf{U} + \mathbf{w}$. In order to simplify the following discussion we choose the x direction as the direction of the main flow: $U_x = U$, $U_y = 0$, $U_z = 0$.

We have a homogeneous shear flow [6] when the main motion has a constant velocity in a given direction and a constant lateral velocity gradient throughout the whole field, e.g., $U_x = U(z)$ and $dU_x/dz = S$, so there is an evident lack of isotropy in the system. Moreover, we have a nonzero turbulence shear stresses tensor, the component $\langle w_x w_z \rangle$ is different from zero and it makes a positive contribution only to $\partial_t \langle w_x^2 \rangle$, resulting in nonisotropy.

A generalization of the "4/5" Kolmogorov equation for anisotropic homogeneous shear flow [7] suggests that the typical scale fixed by the shear intensity is $r_s \sim (\epsilon/S^3)^{1/2}$. With zero shear this scale is infinite, otherwise it has a finite value: below this scale the shear effects are expected to become negligible. The particular question to be answered is: what happens to the scaling laws (4) when r_s falls into the inertial range?

In order to answer this question we perform a direct numerical simulation of a turbulent shear flow, using a code based on the lattice Boltzmann equation (LBE). Let us briefly recall some of the most important characteristics of this kind of algorithm; but for computational details see, for instance, [8–11]. The main idea underlying the application of the LBE algorithm to hydrodynamic's problems is that the Navier-Stokes (NS) equations are independent of the details of the microscopic dynamics that enters only in the determination of the transport coefficients. So we can model the microscopic dynamic in a very simple way (e.g., lattice gas automata or lattice Boltzmann equation) and recover the right hydrodynamic behavior in the macroscopic limit, i.e., the one in which the ratio between the particle's mean free path and the scales over which the macroscopic fields fluctuate, goes to zero. In order to reproduce the right behavior of a three-dimensional (3D) fluid we consider a 4D face centered hypercube with periodic boundary conditions along the fourth dimension. In each node there are 24 links to the nearest neighbors, along which particles move with velocity $\mathbf{c}_i$, $n_i(\mathbf{x},t) = \{0,1\}$ being the occupation number of the ith link at the site $\mathbf{x}$, at the time t. Let us indicate with $N_i(\mathbf{x},t) = \langle n_i(\mathbf{x},t)\rangle$ the average population; the macroscopic fields will be

$$\rho = \sum_i N_i, \quad \mathbf{J}(\mathbf{x},t) \equiv \rho \mathbf{v}(\mathbf{x},t) = \sum_i N_i \mathbf{c}_i.$$

The time evolution of the N_i, and so of the velocity field, is driven by a linearized expression of the Boltzmann collision operator:

$$N_i(\mathbf{x},t+1) - N_i(\mathbf{x},t) = A_{ij}(N_j - N_j^{\mathrm{eq}}) \tag{7}$$

where N_j^{eq} is the equilibrium population and A_{ij} is, as far as the numerical simulation is concerned, a numerical parameter with the right properties (symmetric and cyclic) to simulate the NS equations with the desired values of the transport coefficients.

We simulate a 3D fluid occupying a volume of $V = L^3$ sites with $L = 160$, viscosity $\nu = 0.014$, and obeying to the usual NS equations plus a forcing term $\mathbf{f} = (f_x(z),0,0)$ chosen such that the stationary solution of the NS equations is

$$U_x = A \sin(k_z z), \quad U_y = 0, \quad U_z = 0. \tag{8}$$

$k_z = 8\pi/L$ is the wave vector corresponding to the integral scales, and $A = 0.3$.

Thus the shear has a spatial dependence $S(z) \sim \cos(k_z z)$. We have access to both zones where the shear is maximum and locally homogeneous, and zones where the shear is minimum.

We evaluated v_{rms} as the mean value of $(2/3E)^{1/2}$. The simulations were done at $\mathrm{Re}_\lambda = \lambda v_{\mathrm{rms}}/\nu \sim 40$, with $\lambda \sim 15$ lattice spacings, and the Kolmogorov scale is about 1 lattice spacing wide.

The simulation has advanced 100 000 iterations corresponding to about 25 macroscale eddy turnover times $\tau_0 \sim L/v_{\mathrm{rms}}$: 40 velocity configurations have been saved every 2500 time steps, in order to ensure the statistical independence of the different configurations.

We have evaluated the structure functions $F_n(r)$ up to the tenth order. The mean values of $|\delta_r v|^n$ have been evaluated through time and spatial average at fixed z level:

$$\langle O(\mathbf{r},t)\rangle = \frac{1}{T}\int_0^T dt \frac{1}{L^2}\int dx\, dy\, O(\mathbf{r},t).$$

In Fig. 1(a) we have a log-lot plot of the longitudinal (x direction) structure function $F_6(r)$ against $F_3(r)$, obtained from the velocity fields corresponding to the minimum shear level. The statistical errors on the structure functions are of the order of the data-points size. The dashed curve is the best fit done in the range between the 20th and 30th grid point, and corresponds to a slope of 1.79 in good agreement with other measured values of ζ_6. Every point in the plot corresponds to a grid point and the lattice spacing is $\sim 1\eta$ wide. As we can see the ESS holds as usual until $(4–5)\eta$.

Figure 1(b) shows the same plot but at the maximum shear level. It is quite evident that ESS does not hold. In any case, the slope corresponding to the best fit can be estimated

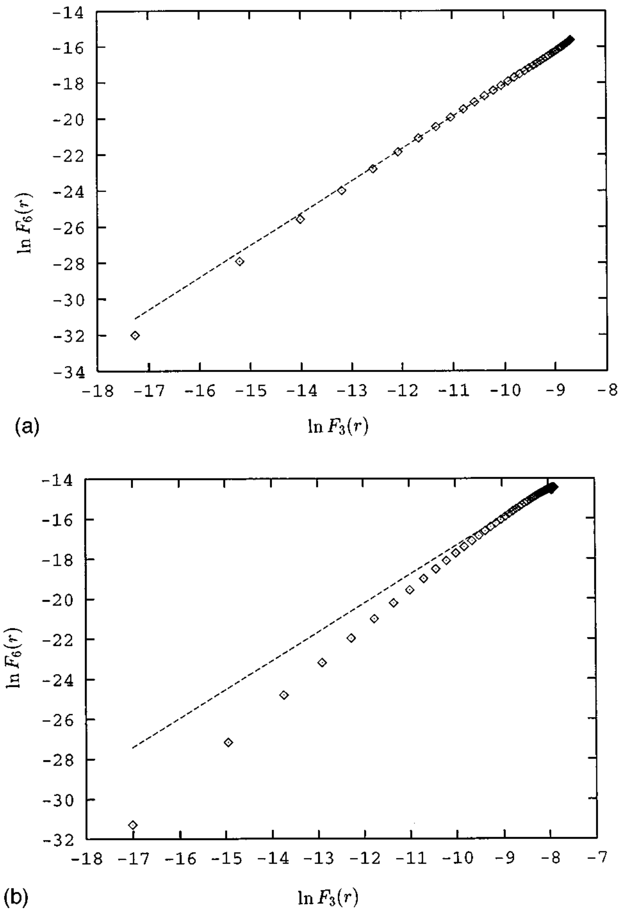

FIG. 1. (a) Log-log plot of $F_6(r)$ against $F_3(r)$ at the minimum shear. The dashed line is the best fit with slope 1.79. Every point in the plot corresponds to a grid point and the lattice spacing is $\sim 1\eta$ wide. All quantities in the plot are expressed in terms of lattice units. (b) The same as in (a) at the maximum shear. The dashed line is the best fit with slope 1.43.

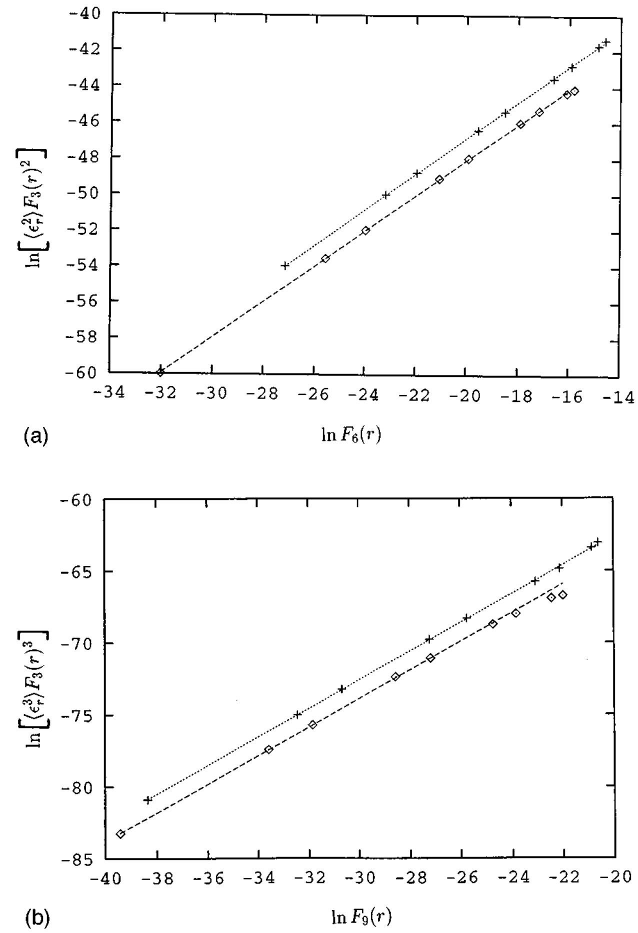

FIG. 2. (a) Log-log plot of Eq. (10) for $n=2$ at the minimum shear (diamonds) and maximum shear (crosses). Plotted data points are at 2, 4, 5, 8, 10, 16, 20, 32, 40 grid points. The dashed lines are the best fits done over these points, corresponding to the slope 0.99. Data referring to the maximum shear has been shifted of one unity. (b) The same as in (a) for $n=3$. The dashed lines are the best fit with slope 0.99.

at about 1.43, quite different from the previous value. Similar results have been obtained for all the others structure functions. In Table I we show the scaling exponents obtained for the even order structure functions.

We can suggest the following explanation for the different scaling behavior in the presence of shear. In our simulations the scale r_s is about 4 lattice spacings at the maximum shear level, so the entire range over which the ESS holds [see Fig. 1(a)] is subjected to the shear effects. Our result clearly shows that the shear completely destroys the ESS.

We now turn our attention to Eq. (3) (RSH). Following [5] we can consider the generalization of RSH by introducing an effective scale $S(r)\equiv\langle\delta_r v^3\rangle/\langle\epsilon_r\rangle = r f(r/\eta)$. Then

TABLE I. Scaling exponents evaluated at the minimum shear (first line), at the maximum shear (second line), and from the She-Leveque [12] model.

	ζ_2	ζ_4	ζ_6	ζ_8	ζ_{10}
min sh	0.70	1.28	1.79	2.25	2.68
max sh	0.76	1.18	1.43	1.56	1.61
SL mod.	0.696	1.279	1.778	2.211	2.593

ESS combined with RSH suggests

$$\frac{\delta_r v^3}{S(r)} \sim \epsilon_r . \tag{9}$$

If Eq. (9) is true, as already verified for experimental data sets referring to homogeneous and isotropic turbulence [5], we expect that the locally averaged dissipation and the structure functions satisfy the following scaling law:

$$\langle \delta_r v^{3n}\rangle \sim \frac{\langle \epsilon_r^n\rangle}{\epsilon^n}\langle \delta_r v^3\rangle^n \tag{10}$$

over a range wider than the inertial one.

Using the data from our simulation, we obtained the results shown in Fig. 2. As we can see the scaling of $\langle\epsilon_r^n\rangle\langle|\delta_r v|^3\rangle^n$ against $\langle\delta_r v^{3n}\rangle$ is well verified in both the zones of maximum and minimum shear with a slope very close to one, for the two values $n=2,3$. This is an extremely interesting result and let us briefly discuss its physical meaning. First of all, one could argue that Eq. (10) is a trivial one because for $r<\eta_k$, ϵ_r is constant and $F_{3n} \propto r^{3n}$, thus the scaling $F_{3n} \propto F_3^n$ is obviously satisfied. Furthermore for r in the inertial range Eq. (10) is certainly verified because $(F_3/\epsilon) \propto r$. However, in principle the proportionality constant of Eq. (10) in the inertial and in the dissipative range could be different; the fact that they have been found equal is not trivial. Moreover, our data refer to a quite low Reynolds number simulation, where the scaling of the structure functions $F_n(r)$ with respect to r is absent. Nevertheless, our generalization of the 1962 Kolmogorov refined similarity hypothesis [Eq. (10)] is well verified, supporting the idea (coming from ESS) that the effective scale $\langle \delta_r v^3 \rangle / \langle \epsilon_r \rangle$ has a crucial role in determining the scaling laws of the structure functions. Last but not least, Eq. (10) holds even when ESS is violated, i.e., regardless of the isotropy conditions of the turbulent flow, showing its universal validity.

Let us summarize the results that have been obtained and suggest a possible interpretation for them and what should be their future developments. First of all, it has been shown that ESS does not hold for anisotropic turbulent flows, according to similar results obtained from experimental data sets of turbulent boundary layers [13], where strong shear effects are expected to appear.

This means that moments of different order show a different dependence from the cutoff scale. This means that the shear affects the function $f_p(r/\eta)$, defined in (5), which is no longer the same for all the orders p. Nevertheless, the scaling law (10) is valid even in the presence of shear and at the smallest scales investigated, suggesting that the scaling law of a generic structure function is related to those of the third one and of the energy dissipation in a universal way, for all analyzed scales, a remarkably nontrivial result.

We think that the investigation of the self-scaling properties of the energy dissipation ϵ_r would deserve more attention, in order to understand how the structure functions of the velocity increments depend on the resolution scale and to explain the ESS violation in shear flows. A deeper analysis of these arguments, together with other numerical and experimental results, will be the subject for further investigation [14,15].

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